values (from a series of computer runs) for the weighting constants: $K_p = 5 \times 10^{-4}$, $K_p' = 5 \times 10^{-6}$, $K_s = 10^{-1}$, and $K_s' = 5 \times 10^{-3}$ with C and β set arbitrarily at 1.50 and 0.50, respectively. For example, in a typical run (for which the initial conditions are $\dot{h}_0 = 500$ fps; $h_0 = 100{,}000$ ft; $\dot{x}_0 = 5000$ fps; $x_0 = 100$ miles, $I_{sp} = 423.5$; and, initial mass $=$ 1500 slugs) the conditions upon reaching an altitude of 2000 ft are: $x = 120$ ft; $\dot{x} = 30$ fps; $\dot{h} = 67$ fps; altitude $=$ 7° from vertical; ΔV = characteristic velocity = 5800 fps; and elapsed time = 367 sec. The time histories of certain variables along this trajectory are shown in Fig. 5. The simulation run presented here was terminated early because of the tendency of $\ddot{x}_C$ computed from Eq. (10) to take on large erroneous values as h approached zero. This difficulty is easily resolved by holding the denominator of Eq. (10) fixed whenever h decreases below a threshold value.

A Method of Calculating Rocket Plume Radiation to the Base Region

C. L. Tien* and M. M. Abu-Romia†
University of California, Berkeley, Calif.

In the design of large booster vehicles, it is recognized that the base regions should be protected against heating by the rocket exhaust plumes. The present work describes an analytical attempt to calculate the radiative energy transfer from rocket exhaust plumes to the base regions by use of idealized physical models. Consider a semi-infinite cylindrical gas body of uniform temperature and composition, emitting and absorbing radiative energy, as shown in Fig. 1. The gas body is separated from a differential area dA by a nonabsorbing medium. No scattering of radiation exists in the system. The spectral apparent emissivity, defined as the ratio of the radiative energy flux to that of a blackbody at the same temperature, is given as[1]

$$\epsilon_\lambda = \frac{1}{\pi} \int_\beta \int_\phi (1 - e^{-A_\lambda S}) \sin\beta \cos\beta d\beta d\phi \qquad (1)$$

where $A_\lambda = a_\lambda r_0$, $S = (s/r_0)$, a_λ is the linear spectral absorption coefficient, r_0 the radius of the cylindrical body; and the path length s is a function of the height of shielding h, the radial distance in the base plane r, the azimuth angle ϕ, and the polar angle β. Thus, $S = S(H,R,\phi,\beta)$ and $\epsilon_\lambda = \epsilon_\lambda(H,R,A_\lambda)$, where $H = (h/r_0)$ and $R = (r/r_0)$.

Equation (1) can be rearranged into a different form as $\epsilon_\lambda = F - \epsilon_{\lambda c}$, where F is the configuration factor:

$$F \equiv \frac{1}{\pi} \int_\beta \int_\phi \sin\beta \cos\beta d\beta d\phi \qquad (2)$$

and $\epsilon_{\lambda c}$ can be regarded as the contribution due to the finite absorption coefficient of the gas body

$$\epsilon_{\lambda c} \equiv \frac{1}{\pi} \int_\beta \int_\phi e^{-A_\lambda S} \sin\beta \cos\beta d\beta d\phi \qquad (3)$$

A more convenient form for the integration of F and $\epsilon_{\lambda c}$ can be accomplished by introducing β', the projection of polar angle β onto the vertical plane, as shown in Fig. 1, where

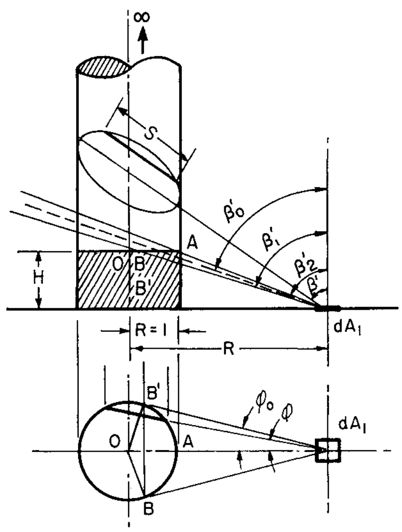

Fig. 1 Semi-infinite cylindrical gas body.

$\tan\beta = \sec\phi \tan\beta'$. Thus, the configuration factor can be written

$$F = \frac{2}{\pi} \int_0^{\beta_0'} \int_0^{\phi_0} \frac{\cos^2\phi \tan\beta' \sec^2\beta'}{(\cos^2\phi + \tan^2\beta')^2} d\beta' d\phi \qquad (4)$$

where

$$\beta_0' = \frac{1}{2}\left[\tan^{-1}\left(\frac{R-1}{H}\right) + \tan^{-1}\left(\frac{R^2-1}{RH}\right)\right] \qquad (5)$$

and $\phi_0 = \sin^{-1}(1/R)$. Equation (4) can be integrated directly by first integrating with respect to $\tan^2\beta'$ instead of β', and the result is

$$F(H,R) = \frac{1}{\pi} \sin\beta_0' \tan^{-1}(\sin\beta_0' \tan\phi_0) \qquad (6)$$

The upper limit β_0', as illustrated in Fig. 1, is being approximated as the arithmetic mean of the two limiting angles β_1' and β_2' for the partially viewed region due to shielding. In the limiting case of $\beta_0' = \pi/2$ (corresponding to $H = 0$), an exact result is obtained for Eq. (6) with no shielding, $F(0,R) = (1/\pi)\sin^{-1}(1/R)$.

The term $\epsilon_{\lambda c}$ defined in Eq. (3) can be expressed from simple geometrical considerations as

$$\epsilon_{\lambda c}(H,R) = \frac{2}{\pi} \int_0^{\beta_0'} \int_0^{\phi_0} e^{-A_\lambda S} \frac{\cos^2\phi \tan\beta' \sec^2\beta'}{(\cos^2\phi + \tan^2\beta')^2} d\beta' d\phi \qquad (7)$$

where the dimensionless path length, as shown in Fig. 1, is given by

$$S(R,\beta',\phi) = \frac{2}{\tan^2\beta'} (1 - R^2 \sin^2\phi)^{1/2} (\cos^2\phi + \tan^2\beta')^{1/2} \qquad (8)$$

The integral in Eq. (7) must be evaluated numerically at different locations specified by H and R. Two asymptotic expressions for $\epsilon_{\lambda c}$, however, can be obtained through direct integration. In the case of no shielding ($H = 0$), the asymptotic expression of ϵ_λ for $A_\lambda \ll 1$ is given as[2]

$$\epsilon_\lambda(0,R) = \frac{4A_\lambda}{\pi}\left[RE_2\left(\frac{\pi}{2}, \frac{1}{R}\right) - \left(\frac{R^2-1}{H}\right) E_1\left(\frac{\pi}{2}, \frac{1}{R}\right)\right] \qquad (9)$$

where E_1 and E_2 are the elliptic integrals of the first and second kinds, respectively. For large R, the spectral apparent emissivity can be expressed as $\epsilon_\lambda(0,R) = A_\lambda/R$. For the case with shielding ($H \neq 0$), the result for $A_\lambda \ll 1$ and $R \gg 1$ can be derived as[2]

$$\epsilon_\lambda(H,R) = (A_\lambda/R) \sin\beta_0' \qquad (10)$$

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* Assistant Professor, Department of Mechanical Engineering.
† Research Assistant, Department of Mechanical Engineering.

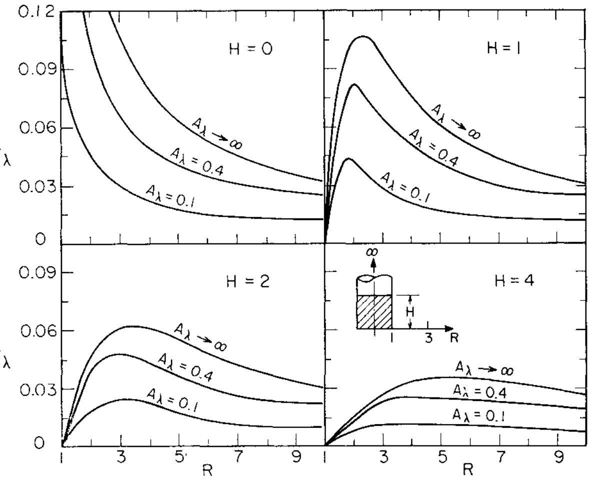

Fig. 2 Spectral apparent emissivity in base plane of a cylindrical gas body.

Numerical results for the spectral apparent emissivity in the base plane of a semi-infinite cylindrical gas body are presented in Fig. 2. They were obtained using a 7090 digital computer. In comparison with the numerical results, the asymptotic expression is found to be a good approximation for $A_\lambda < 0.1$. It should be noted that all the forementioned results are directly applicable to the calculation of the total apparent emissivity if the gray-gas assumption is employed. In that case, A_λ equals A, and ϵ_λ equals ϵ.

Without the assumption of gray gas, the total apparent emissivity can be calculated by utilizing the concept of mean path length.[1] Considering the expression in Eq. (1), a dimensionless mean path length L may be conveniently defined as

$$\epsilon_\lambda \equiv F(1 - e^{-A_\lambda L}) \tag{11}$$

When $A_\lambda \ll 1$, $\epsilon_\lambda \approx FA_\lambda L$, which can be compared with the asymptotic expression of ϵ_λ for $H = 0$ and $A_\lambda \ll 1$, and gives

$$L(0,R) = \frac{4}{\sin^{-1}(1/R)}\left[RE_2\left(\frac{\pi}{2},\frac{1}{R}\right) - \left(\frac{R^2-1}{R}\right)E_1\left(\frac{\pi}{2},\frac{1}{R}\right)\right] \tag{12}$$

A numerical check reveals that the values of ϵ_λ obtained using the mean path length [Eqs. (11) and (12)] agree very well with the numerically calculated results for all values of A_λ. It is also obvious that the agreement becomes exact at two limits, $A_\lambda = 0$ and $A_\lambda \to \infty$. This indicates that the restriction of $A_\lambda \ll 1$ can also be removed, although no rigorous proof has been established here.

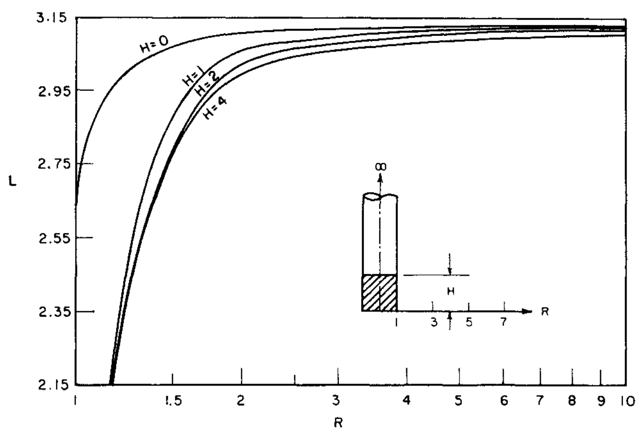

Fig. 3 Dimensionless mean path length for cylindrical gas body.

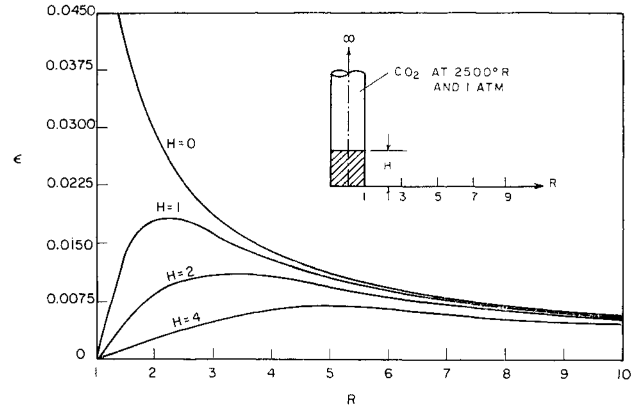

Fig. 4 Apparent emissivities in base plane of a semi-infinite cylinder of carbon dioxide at 2500°R and 1 atm pressure; $r_0 = 2$ ft.

For the case with shielding ($H \neq 0$) a semiempirical equation is suggested as

$$L(H,R) = \frac{4\sin\beta_0'}{\tan^{-1}(\sin\beta_0' \tan\phi_0)}\left[RE_2\left(\frac{\pi}{2},\frac{1}{R}\right) - \left(\frac{R^2-1}{R}\right)E_1\left(\frac{\pi}{2},\frac{1}{R}\right) + \frac{\pi\cos^2\beta_0'}{16R^3}\right] \tag{13}$$

The choice of this equation is based on the consideration that it should reduce to Eq. (12) as $H \to 0$, and to the asymptotic expression of L for $A_\lambda \ll 1$, $R \gg 1$, based on Eq. (10). Again, the comparison between the values of ϵ_λ obtained from Eqs. (11) and (13) and the numerical results from the computer indicates that the condition $A_\lambda \ll 1$ is not necessary in the present case. The values of L as a function of H and R are plotted in Fig. 3. With the expression of L as given in Eq. (13), one can calculate the total apparent emissivity with a given infrared absorption spectrum according to

$$\epsilon(H,R) = F\sum_i(1 - e^{-A_iL})(D_{2i} - D_{1i}) \tag{14}$$

where the functions

$$D_{ni}(\lambda_{ni},T) \equiv \frac{1}{\sigma T^4}\int_0^{\lambda_{ni}} \frac{\pi c_1 d\lambda}{\lambda^5[\exp(c_2/\lambda T) - 1]} \tag{15}$$

are called the relative cumulative spectral radiance of a blackbody and are tabulated as a function of temperature and frequency.[3] The constants c_1 and c_2 are the radiation constants in the Planck distribution.

By using the available absorption measurements for CO_2 and H_2O temperatures,[4,5] numerical calculations have been

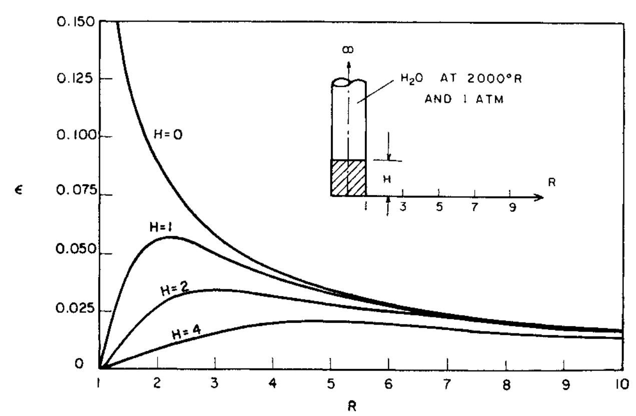

Fig. 5 Apparent emissivities in base plane of a semi-infinite cylinder of water vapor at 2000°R and 1 atm pressure; $r_0 = 2$ ft.

performed, and the results are presented in Figs. 4 and 5. For CO_2 at 2500°F and 1 atm, it is found, by comparing Fig. 4 with corresponding ϵ_λ results (Fig. 2), that the effective wavelength-independent coefficient A is about 0.05 for all four values of H. For H_2O at 2000°R and 1 atm, the value of A is about 0.25 for different values of H.

Formulation for calculating the apparent emissivity of conical gray-gas bodies has been given in Ref. 2, in which numerical results were obtained for conical gas bodies of apex angles 20°, 60°, and 120°.

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Effect of Aerodynamic Drag on Low-Thrust Ascending-Spiral Trajectories

CHONG-HUNG ZEE*

Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

Nomenclature

a = acceleration
A = reference area of satellite
$\bar{c}$ = average exhaust velocity of jet
C_D = drag coefficient (drag force)/$\frac{1}{2}\rho v^2 A$
E = total energy per unit mass
g = gravitational acceleration
k = constant; for k_1, k_2, and k_3, see Eqs. (11), (14), and (15), respectively
$\dot{m}$ = constant mass flow rate
M = mass of satellite, $M \equiv M(t) \equiv M_0 - \dot{m}t$
n = number showing a_0 ($= \dot{m}\bar{c}/M_0$) in terms of g_e
S = semimajor axis
t = time
T = actual time required to escape, including effect of drag
v = velocity
β = reciprocal of scale height
$\gamma(a_0)$ = correction factor from Ref. 4
ϵ = eccentricity of an elliptical orbit
θ = polar angle
μ = gravitational constant of earth
ρ = atmospheric density
ω = angular velocity as defined by $\mu^{1/2}/S^{3/2}$

Subscripts

D = drag
e = at earth's surface
E = escape
ED = escape including effect of drag
0 = initial value at $t = 0$
t = thrust

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* Dynamicist, Dynamic Analysis Advanced Development Group.

FOR low-thrust orbital maneuvers of a low-altitude earth satellite, the perturbation due to aerodynamic forces plays a dominant role in computations of trajectories and propulsion requirements. The case of greatest practical importance in orbital maneuvering is that of tangentially directed thrust, which results not only in the greatest instantaneous rate of energy change, but also in a good approximation to the optimal fuel expenditure.[1] This paper treats ascending spiral trajectories under the influences of constant tangential thrust and aerodynamic drag. The results derived constitute the zero-order approximation to the problem, whereas the first-order approximation will invariably contain oscillatory terms, which, in some similar simpler problems, have been well explored.[2, 3]

Analysis

The velocity of a satellite in an elliptical orbit is

$$v = \left[\frac{\mu(1+\epsilon^2)}{S(1-\epsilon^2)}\right]^{1/2}\left[1 + \frac{\epsilon}{1+\epsilon^2}\cos\theta - \frac{1}{2}\left(\frac{\epsilon}{1+\epsilon^2}\right)^2 \times \cos^2\theta + \ldots\right] \quad (1)$$

If the orbit is nearly circular (ϵ is very small), the higher-order terms in the bracketed series may be neglected, and after averaging over one revolution, the result becomes

$$v = \left[\frac{\mu(1+\epsilon^2)}{S(1-\epsilon^2)}\right]^{1/2} \simeq (\mu/S)^{1/2}(1+\epsilon^2) \quad (2)$$

If the orbit remains nearly circular,

$$v = (\mu/S)^{1/2}(1+\epsilon_0^2) \quad (3)$$

Thus with only a small error, of the order of magnitude of ϵ_0^2, the velocity of a low-altitude earth satellite in a nearly circular orbit can be approximated by

$$v = (\mu/S)^{1/2} \quad (4)$$

Under a low constant tangential thrust, its acceleration is

$$a_t = \dot{m}\bar{c}/M \quad (5)$$

its deceleration due to aerodynamic drag is

$$a_D = C_D A \rho v^2/2M \quad (6)$$

and its rate of change of energy is

$$dE/dt = (a_t - a_D)v = (\dot{m}\bar{c} - C_D A\rho v^2/2)v/M \quad (7)$$

in which, for an ascending spiral trajectory, a_t should be greater than a_D, and

$$E = -\mu/2S \qquad \text{or} \qquad dE/dt = (\mu/2S^2)dS/dt \quad (8)$$

Atmospheric density will be approximated by

$$\rho = \rho_0 \exp[-\beta(S - S_0)] = \rho_0 \exp(\beta S_0)\exp(-\beta S) \quad (9)$$

where β is chosen for best fit in the region where the first part of the spiral trajectory (where drag is significant) is located.

Because $(a_t - a_D)$ is small, the orbit remains nearly circular; thus in combining Eqs. (4, 7, 8, and 9), one obtains

$$\frac{\dot{m}\bar{c}}{M}\left[1 - \frac{C_D A \rho_0 \mu}{2\dot{m}\bar{c}S}\exp(\beta S_0)\exp(-\beta S)\right] = \frac{\mu^{1/2}}{2S^{3/2}}\frac{dS}{dt} \quad (10)$$

After rearranging various terms and expanding the terms in the brackets into series, the result is

$$-\bar{c}\, dM/M = \tfrac{1}{2}\mu^{1/2}S^{-3/2}\{1 + k_1[S\exp(\beta S)]^{-1} + k_1^2[S\exp(\beta S)]^{-2} + \ldots\}dS \quad (11)$$